# Modulational instability of partially coherent signals in electrical transmission lines

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We present an investigation of the modulational instability of partially coherent signals in electrical transmission lines. Starting from the modified Ginzburg-Landau equations and the Wigner-Moyal representation, we derive a nonlinear dispersion relation for the modulational instability. It is found that the effect of signal broadbandness reduces the growth rate of the modulational instability.

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About a decade ago, Marquie *et al.* [1] and Bilbault *et al.* [2] investigated, theoretically and experimentally, the nonlinear propagation of signals in electrical transmission lines. Specifically, they considered a nonlinear network composed of a set of cells containing two linear inductances in series and one in parallel, together with a nonlinear capacitance diode in the shunt branch. It has been shown that the system of equations governing the physics of this network can be reduced to a cubic nonlinear Schrödinger (CNS) equation or a pair of coupled nonlinear Schrödinger (CNLS) equations. Both the CNS and CNLS equations admit modulational instability and the formation of envelope solitons, which have been observed experimentally [1,2].

Recently, Kengne and Liu [3] presented a model for wave propagation on a discrete electrical transmission line based on the modified complex Ginzburg-Landau (MCGL) equation, derived in the small amplitude and long wavelength limit using the standard reductive perturbation technique and complex expansion [4] on the governing nonlinear equations. The MCGL is also referred to as the derivative nonlinear Schrödinger equation or the cubic-quintic Ginzburg-Landau equation. Nonlinear soliton solutions of the MCGL equation have been presented in Ref. [3].

In this brief report, we consider the modulational instabilities of partially coherent electrical pulses that are governed by the MCGL equation [3]

$$i\partial_t u - P\partial_x^2 u - \gamma u - Q_1|u|^4 u - iQ_2|u|^2\partial_x u - iQ_3\partial_x(|u|^2 u) = 0, \tag{1}$$

where $P$, $Q_j$ $(j=1,2,3)$ and $\gamma$ are real transmission line parameters. We note that Eq. (1) has the space-independent harmonic solution $u=u_0\exp(-i\Omega_0 t)$, where $\Omega_0=\gamma+Q_1U_0^4$.

Next, we let $u=[u_0+u_1(t,x)]\exp(-i\Omega_0 t)$ in Eq. (1), where $|u_1|\ll u_0$, and linearize with respect to $u_1$ to obtain

$$i\partial_t u_1 - P\partial_x^2 u_1 - 2Q_1u_0^4(u_1+u_1^*) - iQ_2u_0^2\partial_x u_1 - iQ_3u_0^2\partial_x(2u_1+u_1^*) = 0, \tag{2}$$

where the asterisk denotes the complex conjugate. Separating the perturbation into its real and imaginary parts, according to $u_1=X+iY$, and letting $X,Y\propto\exp(iKx-i\Omega t)$, we obtain from (2) the nonlinear dispersion relation

$$\Omega = -(Q_2+2Q_3)u_0^2K \pm [Q_3^2u_0^4K^2 + (PK^2-4Q_1u_0^4)PK^2]^{1/2}, \tag{3}$$

where $K$ and $\Omega$ are the wave number and the frequency of low-frequency perturbations modulating the carrier signal, respectively. For $\Omega=i\Gamma-(Q_2+2Q_3)u_0^2K$, we obtain the modulational instability growth rate from (3)

$$\Gamma = K[(4PQ_1-Q_3^2)u_0^4 - P^2K^2]^{1/2}, \tag{4}$$

when $PQ_1>0$. We see that the effect of the derivative nonlinearity $Q_3$ is to decrease the instability region, while the higher-order nonlinearity coefficient $Q_1$ tends to increase the instability region. In Fig. 1, the typical structure of the modulational instability growth rate is depicted.

In order to analyze the effects due to partial coherence on the pulse propagation in nonlinear electrical transmission lines, we next introduce the Wigner function, defined as the Fourier transform of the two-point correlation function [5]

$$\rho(t,x,k) = \frac{1}{2\pi}\int d\xi e^{ik\xi}\langle u^*(t,x+\xi/2)u(t,x-\xi/2)\rangle, \tag{5}$$

where the angular brackets denote the ensemble average [7]. The Wigner function defines a generalized phase space

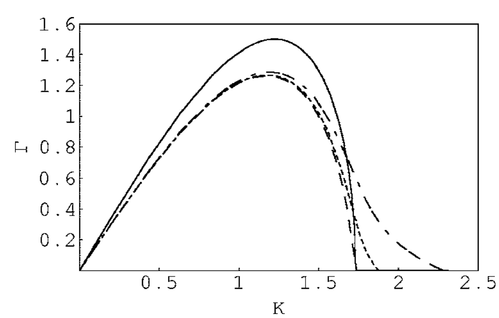


FIG. 1. The typical structure of the modulational instability growth rate $\Gamma$ as a function of the wave number $K$, as given by Eq. (4). The full curve has full coherence ($\Delta=0$), while the dashed curve has a nonzero decoherence as given by the width $\Delta$. The dotted curve has negative values on the parameters $Q_2$ and $Q_3$ and $Q_2=Q_3/4$, while $\Delta$ is still finite, and the dashed-dotted curve has $Q_3=Q_2/4<0$ with a finite $\Delta$.

 

distribution function for quasiparticles, which satisfies the relation

$$I(t,x) \equiv \langle |u(t,x)|^2 \rangle = \int dk\, \rho(t,x,k), \tag{6}$$

where $I$ is the pulse intensity. Applying the time derivative on the definition (5) and using the MCGL equation (1), we obtain [5,6]

$$\partial_t \rho - 2Pk\partial_x \rho - 2Q_1 I^2 \sin\left(\frac{1}{2}\overleftarrow{\partial}_x \vec{\partial}_k\right)\rho - Q_2 I\left[\cos\left(\frac{1}{2}\overleftarrow{\partial}_x \vec{\partial}_k\right)\partial_x \rho - 2\sin\left(\frac{1}{2}\overleftarrow{\partial}_x \vec{\partial}_k\right)k\rho\right] - Q_3\left\{\partial_x\left[I\cos\left(\frac{1}{2}\overleftarrow{\partial}_x \vec{\partial}_k\right)\rho\right] - 2kI\sin\left(\frac{1}{2}\overleftarrow{\partial}_x \vec{\partial}_k\right)\rho\right\} = 0, \tag{7}$$

where the sin and cos operators are defined in terms of their respective Taylor expansion. We note that the $\gamma$ term drops out, since it contains only the phase information for $u$. Equation (7) describes the propagation of partially coherent pulses in nonlinear electrical transmission lines.

We now analyze Eq. (7) for small perturbations, i.e., we let $\rho(t,x,k)=\rho_0(k)+\rho_1 \exp(iKx-i\Omega t)$ and $I(t,x)=I_0 + I_1 \exp(iKx-i\Omega t)$, where $|\rho_1| \ll \rho_0$ and $|I_1| \ll I_0$. Linearizing Eq. (7), we obtain the nonlinear dispersion relation

$$1 = \int dk \left\{ \frac{2Q_1 I_0 + (Q_2 - Q_3)k - \frac{1}{2}(Q_2+Q_3)K}{\Omega + 2PKk + (Q_2+Q_3)KI_0}\rho_{0-} - \frac{2Q_1 I_0 + (Q_2 - Q_3)k + \frac{1}{2}(Q_2+Q_3)K}{\Omega + 2PKk + (Q_2+Q_3)KI_0}\rho_{0+} \right\}, \tag{8}$$

where $\rho_{0\pm}=\rho_0(k \pm K/2)$.

If the background wave function $u_0$ has a partially coherent phase, the corresponding quasiparticle distribution is given by the Lorentzian [8]

$$\rho_0(k) = \frac{I_0}{\pi}\frac{\Delta}{k^2+\Delta^2}, \tag{9}$$

where $\Delta$ is the width of the distribution, giving the degree of decoherence of the pulse intensity. Using the distribution (9) in the general dispersion relation (8), we obtain

$$\Omega = -(Q_2+2Q_3)I_0 K + 2iP\Delta K \pm [Q_3^2 I_0^2 K^2 + (PK^2 - 4Q_1 I_0^2)PK^2 + 2iP(Q_2 - Q_3)I_0 \Delta K^2]^{1/2}. \tag{10}$$

We will assume that $PQ_1 > 0$ in order to make a comparison to the coherent case. With the normalization $\Omega = Q_1 I_0^2 \tilde{\Omega}$, $K=(Q_1/P)^{1/2} I_0 \tilde{K}$, $\Delta=(Q_1/P)^{1/2} I_0 \tilde{\Delta}$, $Q_2=(PQ_1)^{1/2}\tilde{Q}_2$, and $Q_3=Q_2\tilde{Q}_3$, we obtain the dimensionless dispersion relation

$$\Omega = -Q_2(1+2Q_3)K + 2i\epsilon\Delta K \pm [K^4 + (Q_2^2 Q_3^2 - 4)K^2 + 2i\epsilon Q_2(1-Q_3)\Delta K^2]^{1/2}, \tag{11}$$

where we have dropped the tilde on all variables, and $\epsilon = \mathrm{sgn}(P)$. In Fig. 1, we have plotted the normalized growth rate $\Gamma$ as a function of the normalized wave number $K$. We have assumed that $P<0$, i.e., $\epsilon=-1$. When $Q_2=Q_3/4=1/2$ and $\Delta=0$, we obtain the full curve in Fig. 1, while $\Delta=0.1$ gives the dashed curve. For $\Delta=0.1$, but $Q_2=Q_3/4=-1/2$, we obtain the dotted curve in Fig. 1, and $Q_3=Q_2/4=-1/2$ gives the dashed-dotted curve. When $\epsilon=1$, a reduced distribution width $\Delta$ tends to increase the growth rate, which is unphysical.

To summarize, we have examined the modulational instability of partially coherent pulses in nonlinear electrical transmission lines. For this purpose, we have derived a nonlinear dispersion relation from the MCGL equation by using the Wigner-Moyal representation. The nonlinear dispersion relation is analyzed for a Lorentzian equilibrium pulse distribution function. It is found that the growth rate of the modulational instability is reduced due to the consideration of the broadband signals. The present results should help to understand the nonlinear propagation of broadband pulses in electrical transmission lines.

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